AD-A099 338 AIR FORCE GEOPHYSICS LAB HANSCOM AFB MA F/G 20/7
ON THE FIRST AND A RELATED ADIABATIC INVARIANT WITH A FORCE IN --ETC(U)
JUN 80 C W DUBS
UNCLASSIFIED AFGL-TR-80-0193 NL

1 OF 1
AD A
099338

END
DATE
FILMED
6-81
DTIC

AD A099338

LEVEL II

AFGL-TR-80-0193
ENVIRONMENTAL RESEARCH PAPERS, NO. 708

On the First and a Related Adiabatic Invariant With a Force in the Direction of the Velocity

CHARLES W. DUBS

25 June 1980

Approved for public release; distribution unlimited.

SPACE PHYSICS DIVISION PROJECT 7601
AIR FORCE GEOPHYSICS LABORATORY
HANSCOM AFB, MASSACHUSETTS 01731

AIR FORCE SYSTEMS COMMAND, USAF

This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

John Howard

Chief Scientist

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFGL-TR-80-0193
2. GOVT ACCESSION NO.: AD-A099 338
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): ON THE FIRST AND A RELATED ADIABATIC INVARIANT WITH A FORCE IN THE DIRECTION OF THE VELOCITY
5. TYPE OF REPORT & PERIOD COVERED: Scientific. Interim.
6. PERFORMING ORG. REPORT NUMBER: ERP No. 708
7. AUTHOR(s): Charles W. Dubs
8. CONTRACT OR GRANT NUMBER(s):
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Air Force Geophysics Laboratory (PHG), Hanscom AFB, Massachusetts 01731
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 62101F, 7601 15 02
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Geophysics Laboratory (PHG), Hanscom AFB, Massachusetts 01731
12. REPORT DATE: 25 Jun[illegible]80
13. NUMBER OF PAGES: 20
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS (of this report): Unclassified
15a. DECLASSIFICATION DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number):

First adiabatic	Slowing down
Relativistic adiabatic	Charged particle
Protons	Magnetic field
Inhomogeneous	Velocity force
Invariant	Field-geometric

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Is the well known first adiabatic invariant $p_\perp^2/2mB$, $p_\perp$ the perpendicular component of momentum of a charged particle in a magnetic field $\vec{B}$ still an adiabatic invariant if the particle experiences a force in the $\pm\vec{v}$ direction (for example, if it is being slowed down)? No. Is there a related quantity that is? Both a simple and a rigorous proof are given that the closely related quantity $C \equiv \sin^2\alpha/B$, α the pitch angle, is an adiabatic invariant for a particle of any energy in any magnetic field experiencing a force with any v

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified 4095

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. Abstract (Continued)

dependence in the $\pm\vec{v}$ direction. dC/dt is independent of this force, and is the same derived relativistically as not. For the simplest field with the following gradients, the rigorous proof shows C to be constant to second order of the gradient of $B_{\parallel}$ in the direction of $\vec{B}$, and to first order of the gradient of $B_{\parallel}$ perpendicular to $\vec{B}$. The adiabatic invariance of C is useful for analytical treatments of trapped and auroral protons slowing down between nonforward scatterings. Φ but not K is also adiabatically invariant with slowing down.

Contents

On the First and a Related Adiabatic Invariant With a Force in the Direction of the Velocity

1. INTRODUCTION

The adiabatic invariance of $p_{\perp}^2/2mB$, the first adiabatic invariant, $p_{\perp}$ the component of momentum perpendicular to the magnetic field $\vec{B}$ of a charged particle of rest mass m, has been known for three decades. Is this quantity an adiabatic invariant if the particle experiences a force $f(v)\hat{v}$, $\vec{v}$ the particle's velocity, for example, if it is slowing down? No. Is there a related quantity that is? Yes. To the best of the author's knowledge, Dong Lin[1] was the first person to point out that the closely related quantity

$$C = \frac{\sin^2 \alpha}{B} , \qquad (1)$$

α the pitch angle, is an adiabatic invariant for this case. One may reason intuitively that a force in the $\pm\hat{v}$ direction will not change α and therefore not C, but will cause the guiding center to move across the field lines since the radius of gyration is proportional to $v_{\perp}$. Lin[2] and Dubs[3] independently derived the adiabatic invariance of C nonrelativistically for this case by a relatively simple but not very rigorous method, a modification of that by Alfvén and Fälthammar.[4] A rigorous

(Received for publication 24 June 1980)

Because of the large number of references cited above, they will not be listed here. See References, page 21.

nonrelativistic method has also been found.[3] Is C an adiabatic invariant for relativistic particles slowing down? Yes. A modification of the method of Alfvén and Fälthammar[4] that shows this is presented in the next section. This is followed by a rigorous, longer method that results in the time derivative of C. Other adiabatic invariants are then examined briefly. A condensed version of this report has been submitted to a journal.[5]

2. SIMPLE DERIVATION

At any time t_o, choose the origin of a cylindrical coordinate system at the center of gyration of the particle of charge e and rest mass m with the z axis in the direction of $\vec{B}$. Assume a constant magnetic field gradient $\frac{\partial B}{\partial z}$. From $\nabla \cdot \vec{B} = 0$, neglecting $\frac{\partial B_\phi}{\partial \phi}$ and $\frac{\partial B_R}{\partial \phi}$, $B_R = -\frac{R}{2}\frac{\partial B}{\partial z}$. $v_\perp = v_\phi$; ev_ϕ and thus $ep_\perp$ are negative. Let

$$\dot{X} \equiv \frac{dX}{dt} \tag{2}$$

$$\vec{p} = \vec{p}_\parallel + \vec{p}_\perp \tag{3}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \tag{4}$$

$$E = \sqrt{m^2c^4 + p^2c^2} = m\gamma c^2 = \sqrt{m^2c^4 + p_\parallel^2c^2 + p_\perp^2c^2} \tag{5}$$

$$\dot{E} = \frac{p\dot{p}c^2}{E} = m\gamma^3 v\dot{v} = v_\parallel \dot{p}_\parallel + v_\perp \dot{p}_\perp \tag{6}$$

$$\dot{p} = m\gamma^3 v\dot{v}\,\frac{m\gamma c^2}{m\gamma v c^2} = m\gamma^3 \dot{v}. \tag{7}$$

The component of force in the z direction is

$$\dot{p}_\parallel = -ev_\perp B_R + \dot{p}\frac{v_\parallel}{v} = \frac{ev_\perp R}{2}\frac{\partial B}{\partial z} + \dot{p}\frac{v_\parallel}{v} = -\frac{v_\perp}{2}\frac{p_\perp}{B}\frac{\partial B}{\partial z} + m\gamma^3\frac{v_\parallel}{v}\dot{v} \tag{8}$$

---

5. Dubs, C.W. (1980) Adiabatic invariants for a charged particle slowing down, submitted to Journal of Geophysical Research.

since R nearly $= -\frac{p_\perp}{eB}$, the radius of gyration. Substituting this into the expression for $\dot{E}$:

$$m\gamma^3 v\dot{v} = -\frac{v_\perp p_\perp}{2B}\frac{dz}{dt}\frac{\partial B}{\partial z} + m\gamma^3\frac{v_\parallel^2}{v}\dot{v} + v_\perp\dot{p}_\perp \ , \tag{9}$$

$$-m\gamma^3\frac{v^2 - v_\parallel^2}{v}\dot{v} + v_\perp\dot{p}_\perp - \frac{v_\perp p_\perp}{2B}\dot{B} = 0 \ . \tag{10}$$

Multiplying by $\frac{2}{v_\perp p_\perp}$:

$$-2\frac{\gamma^2}{v}\dot{v} + \frac{2}{p_\perp}\dot{p}_\perp - \frac{1}{B}\dot{B} = 0 \ . \tag{11}$$

Substituting $p_\perp = m\gamma v \sin\alpha$:

$$-\frac{2\gamma^2}{v}\dot{v} + \frac{2}{\gamma v}\frac{d(\gamma v)}{dt} + \frac{2}{\sin\alpha}\frac{d\sin\alpha}{dt} - \frac{1}{B}\frac{dB}{dt} = \frac{2}{\sin\alpha}\frac{d\sin\alpha}{dt} - \frac{1}{B}\frac{dB}{dt} = 0 \ , \tag{12}$$

$$\frac{d}{dt}\left[\ln\frac{\sin^2\alpha}{B}\right] = 0 \ , \tag{13}$$

$$\frac{\sin^2\alpha}{B} = \text{constant} \ . \tag{14}$$

3. RIGOROUS DERIVATION

3.1 Simple Magnetic Field

Choose a cylindrical coordinate system as above. Assume a vector potential

$$\vec{A} = \hat{\phi}\frac{bR}{2}\left(1 + \frac{z}{h} + \frac{2R}{3H}\right) \ . \tag{15}$$

Let $b\vec{B}$ be the magnetic field. The normalized magnetic field then is

$$\vec{B} = \frac{1}{b}\nabla\times\vec{A} = -\hat{R}\frac{R}{2h} + \hat{z}\left(1 + \frac{z}{h} + \frac{R}{H}\right) \ . \tag{16}$$

($\vec{B}$ is normalized from here through Eq. (63).) The constants b, h, and H are seen to be the field at the origin and the B_z doubling distances respectively.

$$L = -mc^2 \sqrt{1 - v^2/c^2} + e\vec{A} \cdot \vec{v} \quad , \quad \vec{v} = \hat{R}\dot{R} + \hat{\phi}R\dot{\phi} + \hat{z}\dot{z} \quad . \tag{17}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} + Q_i \quad . \tag{18}$$

Q_i represents the generalized component of the drag force $f(v)\hat{v}$. (See Section 3.2.)

$$\frac{dE}{dt} = m\gamma^3 v\dot{v} = \vec{f} \cdot \vec{v} = f(v)v \quad . \tag{19}$$

So,

$$f(v) = m\gamma^3 \dot{v} \quad . \tag{20}$$

(The latter equals $\dot{p}$, as it must.) Only the R and z Lagrange equations are needed:

$$\begin{cases} \frac{d}{dt}[m\gamma\dot{R}] = m\gamma R\dot{\phi}^2 + ebR\dot{\phi}(\ ) + m\gamma^3\dot{v}\,\frac{\dot{R}}{v} & (21a) \\[2ex] \frac{d}{dt}[m\gamma\dot{z}] = \frac{ebR^2\dot{\phi}}{2h} + m\gamma^3\dot{v}\,\frac{\dot{z}}{v} \quad , & (21b) \end{cases}$$

which yield

$$\begin{cases} \ddot{R} = R\dot{\phi}^2 + \frac{ebR\dot{\phi}}{m\gamma}(\ ) + \frac{\dot{v}\dot{R}}{v} & (22a) \\[2ex] \ddot{z} = \frac{ebR^2\dot{\phi}}{2m\gamma h} + \frac{\dot{v}\dot{z}}{v} \quad . & (22b) \end{cases}$$

The equations are now partially normalized with $\epsilon = \frac{R_o}{h}$, $E = \frac{R_o}{H}$, $R = R_o\rho$, $z = R_o\zeta$, $\vec{v} = R_o\vec{u}$, where R_o is the initial value of R and of the gyroradius. (Note that E from here to the end of Section 3.1 means this instead of energy.)

$$(\ ) \equiv \left(1 + \frac{z}{h} + \frac{R}{H}\right) = (1 + \epsilon\zeta + E\rho) \tag{23}$$

$$\ddot{\rho} = \rho\dot{\phi}^2 + \frac{eb\rho\dot{\phi}}{m\gamma}(\ ) + \frac{\dot{u}\dot{\rho}}{u} \tag{24a}$$

$$\ddot{\zeta} = \frac{\epsilon e b \rho^2 \dot{\phi}}{2m\gamma} + \frac{\dot{u}\dot{\zeta}}{u} . \tag{24b}$$

$$D \equiv b\frac{dC}{dt} - \frac{d}{dt}\left[\frac{1 - \cos^2\alpha}{B}\right] \frac{d}{dt}\left[\frac{1}{B} - \frac{(\vec{u}\cdot\vec{B})^2}{u^2 B^3}\right] \tag{25}$$

$$\vec{B} = -\hat{R}\,\frac{\epsilon\rho}{2} + \hat{z}\,(\;) , \tag{26}$$

$$B^2 = (\;)^2 + \frac{\epsilon^2\rho^2}{4} , \tag{27}$$

$$\vec{u} = \hat{R}\dot{\rho} + \hat{\phi}\rho\dot{\phi} + \hat{z}\dot{\zeta} , \tag{28}$$

$$\vec{u}\cdot\vec{B} = \dot{\zeta}\,(\;) - \frac{\epsilon}{2}\rho\dot{\rho} . \tag{29}$$

$$\begin{aligned} D = \frac{d}{dt}\Bigg\{&\left[(\;)^2 + \frac{\epsilon^2\rho^2}{4}\right]^{-1/2} - \left[\dot{\zeta}\,(\;) - \frac{\epsilon}{2}\rho\dot{\rho}\right] u^{-2}\left[(\;)^2 + \frac{\epsilon^2\rho^2}{4}\right]^{-3/2}\Bigg\} \\ &- \frac{1}{B^3}\left[(\;)(\epsilon\dot{\zeta} + E\dot{\rho}) + \frac{\epsilon^2}{4}\rho\dot{\rho}\right] - \frac{2}{u^2B^3}\left[\dot{\zeta}\,(\;) - \frac{\epsilon}{2}\rho\dot{\rho}\right] \\ &\left[\ddot{\zeta}\,(\;) + \dot{\zeta}(\epsilon\dot{\zeta} + E\dot{\rho}) - \frac{\epsilon}{2}(\dot{\rho}^2 + \rho\ddot{\rho})\right] \\ &+ \frac{2\dot{u}}{u^3B^3}\left[\dot{\zeta}\,(\;) - \frac{\epsilon}{2}\rho\dot{\rho}\right]^2 + \frac{3}{u^2B^5}\left[\dot{\zeta}\,(\;) - \frac{3}{2}\rho\dot{\rho}\right]^2\left[(\;)(\epsilon\dot{\zeta} + E\dot{\rho}) + \frac{\epsilon^2}{4}\rho\dot{\rho}\right] . \end{aligned} \tag{30}$$

Let M be $u^2 B^3$ times the middle two of these four terms. Use the two equations of motion to eliminate $\ddot{\rho}$ and $\ddot{\zeta}$.

$$\begin{aligned} M = -2\left[\dot{\zeta}\,(\;) - \frac{\epsilon}{2}\rho\dot{\rho}\right]\Bigg[&\frac{\epsilon e b \rho^2\dot{\phi}}{2m\gamma}\,(\;) + \frac{\dot{\zeta}\dot{u}}{u}\,(\;) + \epsilon\dot{\zeta}^2 + E\dot{\rho}\dot{\zeta} - \frac{\epsilon}{2}\dot{\rho}^2 - \frac{\epsilon}{2}\rho^2\dot{\phi}^2 \\ &- \frac{\epsilon e b\rho^2\dot{\phi}}{2m\gamma}\,(\;) - \frac{\epsilon\rho\dot{\rho}\dot{u}}{2u} - \frac{\dot{\zeta}\dot{u}}{u}\,(\;) + \frac{\epsilon\rho\dot{\rho}\dot{u}}{2u}\Bigg] \end{aligned} \tag{31a}$$

$$= \left[\dot{\zeta}\,(\;) - \frac{\epsilon}{2}\rho\dot{\rho}\right]\left[\epsilon(u^2 - 3\dot{\zeta}^2) - 2E\dot{\rho}\dot{\zeta}\right] . \tag{31b}$$

Note that all of the relativistic terms and all of the terms containing the change in the particle's speed cancel. So, D is the same with or without slowing down and with or without being derived relativistically. The result then is:

$$
\begin{aligned}
D = \frac{1}{u^2 B^5} \Bigg\{ & -\frac{3\epsilon^2}{4} \rho \Big[\dot{\rho}(u^2 + \dot{\zeta}^2)(1 + \epsilon\zeta + E\rho)^2 + \epsilon\rho\dot{\zeta}^3(1 + \epsilon\zeta + E\rho) \\
& + \frac{\epsilon^2}{4}\rho^2\dot{\rho}(\rho^2\dot{\phi}^2 - \dot{\zeta}^2)\Big] + E\dot{\rho}\Big[-(\dot{\rho}^2 + \rho^2\dot{\phi}^2)(1 + \epsilon\zeta + E\rho)^3 \\
& - 2\epsilon\rho\dot{\rho}\dot{\zeta}(1 + \epsilon\zeta + E\rho)^2 + \frac{\epsilon^2}{4}\rho^2(2\dot{\rho}^2 - \rho^2\dot{\phi}^2 - 3\dot{\zeta}^2)(1 + \epsilon\zeta + E\rho) \\
& + \frac{\epsilon^3}{4}\rho^3\dot{\rho}\dot{\zeta}\Big]\Bigg\} \quad . \qquad (32)
\end{aligned}
$$

Note that, for $E = 0$ ($H = \infty$), D contains terms only in ϵ^2, ϵ^3, and ϵ^4 divided by $u^2 B^5$, and, for $\epsilon = 0$ ($h = \infty$), D contains terms only in E, E^2, E^3, and E^4 divided by $u^2 B^5$. Alternatively, $D = 0(\epsilon^2 E^0) + 0(\epsilon^0 E)$.

3.2 General Magnetic Field

Which of these results hold for any magnetic field? Repeat the last section, but with

$$\vec{A} = \frac{bR}{2}\hat{\phi} + \epsilon b\vec{a}(R, \phi, z) \quad , \quad \vec{a} = \vec{a}' + \nabla U \quad , \qquad (33)$$

where $\vec{a}'$ is an arbitrary function of R, ϕ, and z except that $\nabla \times \vec{a}'$ vanishes at $R = z = 0$. Then $b\vec{B} = \nabla \times \vec{A}$ is completely arbitrary, regardless of what function is chosen for the scalar U. To simplify the algebra, choose

$$U = -\int a'_R \, dR \quad , \qquad (34)$$

holding ϕ and z constant. Then $a_R = 0$, and the normalized field is

$$\vec{B} = \frac{1}{b}\nabla\times\vec{A} = \hat{z} + \epsilon\nabla\times\vec{a} = \hat{z} + \epsilon\left[\hat{R}\left(\frac{1}{R}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right) + \hat{\phi}\left(-\frac{\partial a_z}{\partial R}\right) + \hat{z}\,\frac{1}{R}\frac{\partial(Ra_\phi)}{\partial R}\right]$$

$$\equiv \hat{R}\epsilon\beta_R + \hat{\phi}\epsilon\beta_\phi + \hat{z}(1 + \epsilon\beta_z) \quad . \qquad (35)$$

$$\beta_R = \frac{1}{R}\frac{\partial a_z}{\partial \phi} - \frac{\partial a_\phi}{\partial z} \quad , \tag{36}$$

$$\beta_\phi = -\frac{\partial a_z}{\partial R} \quad , \tag{37}$$

$$\beta_z = \frac{1}{R}\frac{\partial (R a_\phi)}{\partial R} \quad . \tag{38}$$

As in the last section,

$$\frac{dE}{dt} = \vec{f}\cdot\vec{v} = m\gamma^3\,\dot{v}\left(\hat{R}\,\frac{\dot{R}}{v} + \hat{\phi}\,\frac{R\dot{\phi}}{v} + \hat{z}\,\frac{\dot{z}}{v}\right)\cdot\vec{v} \quad , \quad \vec{v} = \hat{R}\,\dot{R} + \hat{\phi}\,R\,\dot{\phi} + \hat{z}\,\dot{z} \quad . \tag{39}$$

From mechanics (for example, Goldstein[6]), the generalized drag force is $Q_i = \vec{f}\cdot\frac{\partial \vec{v}}{\partial \dot{q}_i}$. So

$$Q_R = m\gamma^3\,\frac{\dot{v}\,\dot{R}}{v} \quad , \tag{40}$$

$$Q_\phi = m\gamma^3\,\frac{\dot{v}\,R^2\dot{\phi}}{v} \quad , \tag{41}$$

$$Q_z = m\gamma^3\,\frac{\dot{v}\,\dot{z}}{v} \quad . \tag{42}$$

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} + e\vec{A}\cdot\vec{v}$$

$$= -mc^2\sqrt{1-\frac{\dot{R}^2+R^2\dot{\phi}^2+\dot{z}^2}{c^2}} + eb\left[\left(\frac{R^2}{2}+\epsilon R a_\phi\right)\dot{\phi} + \epsilon a_z\dot{z}\right] \quad . \tag{43}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) = \frac{\partial L}{\partial R} + Q_R \quad . \tag{44}$$

---

6. Goldstein, H. (1950) Classical Mechanics, Addison-Wesley Press, Inc., p. 22.

$$\frac{d}{dt}(m\gamma\dot{R}) = m\gamma R\dot{\phi}^2 + eb\left[R\dot{\phi} + \epsilon\left(\frac{\partial(Ra_\phi)}{\partial R}\dot{\phi} + \frac{\partial a_z}{\partial R}\dot{z}\right)\right] + m\gamma^3\frac{\dot{v}\dot{R}}{v} \quad . \tag{45}$$

$$m\gamma\ddot{R} + m\gamma^3\frac{v\dot{v}}{c^2}\dot{R} = m\gamma R\dot{\phi}^2 + eb\left[R\dot{\phi} + \epsilon\left(R\dot{\phi}\beta_z - \dot{z}\beta_\phi\right)\right] + m\gamma^3\frac{\dot{v}\dot{R}}{v} \quad . \tag{46}$$

$$\ddot{R} = R\dot{\phi}^2 + \frac{eb}{m\gamma}\left[R\dot{\phi} + \epsilon(R\dot{\phi}\beta_z - \dot{z}\beta_\phi)\right] + \frac{\dot{v}\dot{R}}{v} \quad . \tag{47}$$

In general, the equation for $R\ddot{\phi}$ is needed.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi} + Q_\phi \quad . \tag{48}$$

$$\frac{d}{dt}\left[m\gamma R^2\dot{\phi} + eb\left(\frac{R^2}{2} + \epsilon Ra_\phi\right)\right] = eb\epsilon\left(\frac{\partial a_\phi}{\partial\phi}R\dot{\phi} + \frac{\partial a_z}{\partial\phi}\dot{z}\right) + m\gamma^3\frac{\dot{v}R^2\dot{\phi}}{v} \quad . \tag{49}$$

$$m\gamma R^2\ddot{\phi} + 2m\gamma R\dot{R}\dot{\phi} + m\gamma^3\frac{v\dot{v}}{c^2}R^2\dot{\phi} + eb\left(R\dot{R} + \epsilon\frac{d(Ra_\phi)}{dt}\right)$$

$$= \epsilon eb\left(\frac{\partial a_\phi}{\partial\rho}R\dot{\phi} + \frac{\partial a_z}{\partial\phi}\dot{z}\right) + m\gamma^3\frac{\dot{v}R^2\dot{\phi}}{v} \quad . \tag{50}$$

$$R\ddot{\phi} = -2\dot{R}\dot{\phi} + \frac{eb}{m\gamma}\left[-\dot{R} + \frac{\epsilon}{R}\left(\frac{\partial a_\phi}{\partial\phi}R\dot{\phi} + \frac{\partial a_z}{\partial\phi}\dot{z}\right) - \frac{d(Ra_\phi)}{dt}\right] + \frac{\dot{v}R\dot{\phi}}{v} \tag{51}$$

$$R\ddot{\phi} = -2\dot{R}\dot{\phi} + \frac{eb}{m\gamma}\left[-\dot{R} + \epsilon\left(\dot{z}\beta_R - \dot{R}\beta_z\right)\right] + \frac{\dot{v}R\dot{\phi}}{v} \quad . \tag{52}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{z}}\right) = \frac{\partial L}{\partial z} + Q_z \quad . \tag{53}$$

$$\frac{d}{dt}(m\gamma\dot{z} + eb\epsilon a_z) = eb\epsilon\left(\frac{\partial a_\phi}{\partial z}R\dot{\phi} + \frac{\partial a_z}{\partial z}\dot{z}\right) + m\gamma^3\frac{\dot{v}\dot{z}}{v} \quad . \tag{54}$$

$$m\gamma\ddot{z} + m\gamma^3\frac{v\dot{v}}{c^2}\dot{z} + \epsilon eb\frac{da_z}{dt} = \epsilon eb\left(\frac{\partial a_\phi}{\partial z}R\dot{\phi} + \frac{\partial a_z}{\partial z}\dot{z}\right) + m\gamma^3\frac{\dot{v}\dot{z}}{v} \quad . \tag{55}$$

$$\ddot{z} = \epsilon \frac{e\,b}{m\gamma}\left(\frac{\partial a_\phi}{\partial z} R\dot{\phi} + \frac{\partial a_z}{\partial z}\dot{z} - \frac{\partial a_z}{\partial R}\dot{R} - \frac{\partial a_z}{\partial \phi}\dot{\phi} - \frac{\partial a_z}{\partial z}\dot{z}\right) + \frac{\dot{v}\dot{z}}{v} \tag{56}$$

$$\ddot{z} = \epsilon \frac{e\,b}{m\gamma}(-R\dot{\phi}\beta_R + \dot{R}\beta_\phi) + \frac{\dot{v}\dot{z}}{v} \quad . \tag{57}$$

$$B^2 = \epsilon^2(\beta_R^2 + \beta_\phi^2) + (1 + \epsilon\beta_z)^2 . \qquad \vec{v}\cdot\vec{B} = \dot{z} + \epsilon(\dot{R}\beta_R + R\dot{\phi}\beta_\phi + \dot{z}\beta_z) \ , \tag{58}$$

$$D = \frac{d}{dt}\left\{(B^2)^{-1/2} - [\dot{z} + \epsilon(\dot{R}\beta_R + R\dot{\phi}\beta_\phi) + \dot{z}\beta_z]^2\, v^{-2}\,(B^2)^{-3/2}\right\}$$

$$= -\frac{(1+\epsilon\beta_z)\,\epsilon\dot{\beta}_z + \epsilon^2(\beta_R\dot{\beta}_R + \beta_\phi\dot{\beta}_\phi)}{B^3} - \frac{2[\ \ ]\,\epsilon(\dot{R}\dot{\beta}_R + \dot{R}\dot{\phi}\beta_\phi + R\dot{\phi}\dot{\beta}_\phi + \dot{z}\dot{\beta}_z)}{v^2B^3}$$

$$- \frac{2[\ \ ]\{\ddot{z}(1+\epsilon\beta_z) + \epsilon\ddot{R}\beta_R + \epsilon R\ddot{\phi}\beta_\phi\}}{v^2B^3} - \frac{[\ \ ]^2(-2\dot{v})}{v^3B^3}$$

$$- \frac{[\ \ ]^2(-3)\{(1+\epsilon\beta_z)\,\epsilon\dot{\beta}_z + \epsilon^2(\beta_R\dot{\beta}_R + \beta_\phi\dot{\beta}_\phi)\}}{v^2B^5} \quad , \tag{59}$$

where [ ] is $\dot{z} + \epsilon(\dot{R}\beta_R + R\dot{\phi}\beta_\phi + \dot{z}\beta_z)$. Using Eqs. (47), (52), and (57), $\frac{1}{2}v^2B^3$ times the third and fourth terms is

$$-[\ \ ]\left\{\epsilon\frac{e\,b}{m\gamma}(-R\dot{\phi}\beta_R + \dot{R}\beta_\phi)(1+\epsilon\beta_z) + \frac{\dot{v}\dot{z}}{v}(1+\epsilon\beta_z) + \epsilon R\dot{\phi}^2\beta_R\right.$$

$$+ \epsilon\frac{e\,b}{m\gamma}(R\dot{\phi}\beta_R + \epsilon R\dot{\phi}\beta_R\beta_z - \epsilon\dot{z}\beta_R\beta_\phi) + \epsilon\frac{\dot{v}\dot{R}}{v}\beta_R - 2\epsilon\dot{R}\dot{\phi}\beta_\phi$$

$$+ \epsilon\frac{e\,b}{m\gamma}(-\dot{R}\beta_\phi + \epsilon\dot{z}\beta_R\beta_\phi - \epsilon\dot{R}\beta_\phi\beta_z) + \epsilon\frac{\dot{v}R\dot{\phi}}{v}\beta_\phi$$

$$\left. - \frac{\dot{v}\dot{z}}{v} - \epsilon\left(\frac{\dot{v}\dot{R}}{v}\beta_R + \frac{\dot{v}R\dot{\phi}}{v}\beta_\phi + \frac{\dot{v}\dot{z}}{v}\beta_z\right)\right\}$$

$$= -[\ \ ]\,\epsilon\{R\dot{\phi}^2\beta_R - 2\dot{R}\dot{\phi}\beta_\phi\} \quad . \tag{60}$$

Note two things: First, the $\dot{v}$ terms cancel, so D is unchanged by forces in the $\pm\vec{v}$ direction and thus of any processes producing only slowing down. Second, the terms containing γ cancel, so D is the same with a relativistic derivation as it is with a nonrelativistic one. So

$$D = - \frac{(1 + \epsilon \beta_z)\, \epsilon \dot{\beta}_z + \epsilon^2 (\beta_R \dot{\beta}_R + \beta_\phi \dot{\beta}_\phi)}{B^3}$$

$$- \frac{2[\dot{z} + \epsilon(\dot{R} \beta_R + R\dot{\phi} \beta_\phi + \dot{z} \beta_z]\, \epsilon \{\dot{R} \dot{\beta}_R + R\dot{\phi}^2 \beta_R - \dot{R}\dot{\phi} \beta_\phi + R\dot{\phi} \dot{\beta}_\phi + \dot{z} \dot{\beta}_z\}}{v^2 B^3}$$

$$+ \frac{3[\dot{z} + \epsilon(\dot{R} \beta_R + R\dot{\phi} \beta_\phi + \dot{z} \beta_z]^2 \{(1 + \epsilon \beta_z) \epsilon \dot{\beta}_z + \epsilon^2 (\beta_R \dot{\beta}_R + \beta_\phi \dot{\beta}_\phi)\}}{v^2 B^5} \quad . \qquad (61)$$

$$D = -2 \epsilon \dot{z}\, \frac{\dot{R} \dot{\beta}_R + R\dot{\phi}^2 \beta_R - \dot{R}\dot{\phi} \beta_\phi + R\dot{\phi} \dot{\beta}_\phi}{v^2} + 0\,(\epsilon^2) \quad . \qquad (62)$$

Thus, in general, D is of first order in ϵ.

Perhaps the simplest way to see that D is independent of slowing down processes is as follows.

$$D = \frac{d}{dt}\left[\frac{1}{B} - \left(\frac{\hat{B}}{B^{1/2}} \cdot \hat{v}\right)^2\right] = \frac{d}{dt}\left(\frac{1}{B}\right) - 2\left(\frac{\hat{B}}{B^{1/2}} \cdot \hat{v}\right)\left[\frac{d}{dt}\left(\frac{\hat{B}}{B^{1/2}}\right) \cdot \hat{v} + \frac{\hat{B}}{B^{1/2}} \cdot \frac{d\hat{v}}{dt}\right] \quad . \qquad (63)$$

Since $\vec{B}$ (unnormalized from here on) is a function only of R, ϕ, and z, the only thing in this expression which could depend on slowing down is $\frac{d\hat{v}}{dt}$.

$$\frac{d\vec{p}}{dt} = m\gamma v \frac{d\hat{v}}{dt} + m \frac{d(\gamma v)}{dt} \hat{v} = e\vec{v} \times \vec{B} + f\hat{v} \quad . \qquad (64)$$

Dotting $\hat{v}$ into this:

$$f = m \frac{d(\gamma v)}{dt} \quad , \qquad (65)$$

since $\frac{d\hat{v}}{dt}$ must be perpendicular to $\hat{v}$. So

$$\frac{d\hat{v}}{dt} = \frac{e\vec{v}\times\vec{B}}{m\gamma v} \quad , \tag{66}$$

the same as it would without slowing down. Since this is the only term in D containing γ but it contributes nothing to D, the latter is seen again to be independent of whether or not it is derived relativistically.

If there be a component of drag force perpendicular to $\vec{v}$, write the drag force as $\vec{f} = f_v\hat{v} + f_n\hat{n}$, $\hat{n}\cdot\hat{v} = 0$. Carrying out the same procedure,

$$\frac{d\hat{v}}{dt} = \frac{e\vec{v}\times\vec{B} + f_n\hat{n}}{m\gamma v} \quad , \tag{67}$$

and

$$\hat{B}\cdot\frac{d\hat{v}}{dt} = \frac{f_n}{m\gamma v}\,\hat{B}\cdot\hat{n} \tag{68}$$

instead of zero in the expression for D.

4. OTHER ADIABATIC INVARIANTS

More generally, consider a purely field-geometric quantity, $g(B,\mu)$, $\mu = \cos\alpha$.

$$\frac{dg}{dt} = \frac{\partial g}{\partial B}\frac{dB}{dt} + \frac{\partial g}{\partial\mu}\left[\frac{d\hat{B}}{dt}\cdot\hat{v} + \hat{B}\cdot\frac{d\hat{v}}{dt}\right] \quad .$$

By the same reasoning (p. 14, last paragraph), g is seen to be independent of slowing down and of relativity. Further, consider

$$G = \int_{s_m}^{s'_m} g(B,\mu)\,ds \quad ,$$

where s is the distance along a field line quite close to the locus of guiding centers, and s_m and s'_m are mirror point values. For $C = \frac{1}{B_m}$ = const., B_m the mirror point value of B, s_m and s'_m are constant. Then

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

where τ_b is the bounce period.

$C(E_\perp, s_m, s_m')$ is an example of [illegible] I, and Φ, the third adiabatic invariant,

$$I = \int^{s_m'} \left[1 - \frac{B(s)}{E_\perp} \right]^{1/2} ds \; [illegible]$$

Thus, if these quantities are invariant with no slowing down, the [illegible] shows them to be invariant with slowing down.

Note that, in general, M, J (=2pI) and even E cannot be adiabatic [illegible] with slowing down. For example, suppose that:

$$\vec{A} = \hat{\phi}\, \frac{bB}{2} \left[1 + \frac{z^2}{h^2} \right] , \quad [illegible]$$

the particle is slow enough to neglect relativistic effects, and that C is adiabatically invariant. Then:

$$\frac{1}{b}\vec{B} = -\hat{R}\,\frac{Rz}{h^2} + \hat{z}\left[1 + \frac{z^2}{h^2}\right] .$$

A treatment like that in Section 3.1 leads to

$$\ddot{z} = \frac{e\,b\,R^2\dot{\phi}\,z}{mh^2} - \nu\dot{z} .$$

Suppose that at $t = 0$: $v = v_o$, $z = 0$, $\dot{z} = \dot{z}_o$, and $R = R_o$, the radius of gyration.

$$f = m\,\frac{dv}{dt} = -m\nu v ,$$

so

$$v = v_o\,e^{-\nu t} .$$

$$R = \left|\frac{mv}{eB}\right| , \quad \dot{\phi} = -\frac{eB}{m} , \quad R^2\dot{\phi} = -\frac{mCv^2}{e} = -\frac{mCv_o^2}{e}\,e^{-2\nu t} .$$

Therefore,

$$\ddot{z} + \nu\dot{z} + k^2 e^{-2\nu t} z = 0 , \quad k^2 = \frac{bCv_o^2}{h^2} .$$

The solution is:

$$z = \frac{\dot{z}_o}{k}\sin\left[\frac{k}{\nu}(1 - e^{-\nu t})\right] , \quad \dot{z} = \dot{z}_o\,e^{-\nu t}\cos\left[\frac{k}{\nu}(1 - e^{-\nu t})\right] .$$

$$J = \int_{1\ \mathrm{cycle}} p\,ds = \int_{1\ \mathrm{period}} p\,\dot{z}\,dt' = m\int_{t_1}^{t_2}\dot{z}^2\,dt' = m\dot{z}_o^2\int_{t_1}^{t_2} dt'\,e^{-2\nu t'}$$

$$\cos^2\left[\frac{k}{\nu}(1 - e^{-\nu t'})\right] .$$

$$t_1 = t - \frac{1}{\nu}\ln\left(1 + \frac{\pi\nu}{k}e^{\nu t}\right) , \quad t_2 = t - \frac{1}{\nu}\ln\left(1 - \frac{\pi\nu}{k}e^{\nu t}\right) .$$

Let

$$\psi = \frac{k}{\nu}\,(1 - e^{-\nu t'}) \quad .$$

Then

$$J = \frac{m\dot{z}_o^2}{k} \int_{\psi_1}^{\psi_2} \left(1 - \frac{\nu\psi}{k}\right) \cos^2\psi \, d\psi \quad , \quad \psi_1 = \frac{k}{\nu}(1 - e^{-\nu t}) - \pi \quad ,$$

$$\psi_2 = \frac{k}{\nu}(1 - e^{-\nu t}) + \pi \quad .$$

Integrating:

$$J = \frac{\pi m\dot{z}_o^2}{k} \left\{ e^{-\nu t} + \frac{\nu}{2k} \sin\left[\frac{2k}{\nu}\,(1 - e^{-\nu t})\right]\right\} \quad .$$

$$M = \frac{p_\perp^2}{2mB} = \frac{m v_o^2 C}{2}\, e^{-2\nu t} \quad .$$

$$K = \frac{J}{2\sqrt{2mM}} = \frac{\pi \dot{z}_o^2}{2kv_o\sqrt{C}} \left\{1 + \frac{\nu}{2k}\, e^{\nu t} \sin\left[\frac{2k}{\nu}\,(1 - e^{-\nu t})\right]\right\} \quad .$$

All three of these quantities are seen to vary with time. So, even though they are adiabatic invariants with parallel components of force, they are not with slowing down. K will be essentially invariant for $t \ll \nu^{-1}$, but increases exponentially with t at large times unless k/ν is a half integer times π.

5. CONCLUSIONS

The first adiabatic invariant, $p_\perp^2/2mB$, is not invariant if the particle is slowing down. The closely related quantity $C = \frac{2\pi m^2 \omega}{B}$, however, is an adiabatic invariant for any energy particle in any magnetic field, even with an additional force of the form $f(v)\hat{v}$. This is not surprising since a force in the $\pm\hat{v}$ direction should not change ω and thus not C, although it would cause the center of gyration to move across the field (toward the particle if it is being slowed down) since the gyroradius is proportional to $v_\perp$. C is not invariant with a parallel component of electric field

present. $D \equiv b\frac{dC}{dt}$, b a constant, is shown to be independent of f and thus of slowing down for all magnetic fields. It is also shown to be the same whether derived relativistically or not. In general, D is zero to the first order of the gradients. For the simplest magnetic field with a parallel gradient of B_z, $\frac{b}{h}$, and a perpendicular gradient of B_z, $\frac{b}{H}$, $D = 0(\epsilon^2 E^0) + 0(\epsilon^0 E)$, where $\epsilon = R_o/h$, $E = R_o/H$, R_o is the initial gyroradius, and $\vec{B} = b\hat{z}$ at the origin.

The adiabatic invariance of C with a force on the particle in the $\pm\vec{v}$ direction, among other cases, is applicable to trapped and auroral protons between nonforward scatterings. Examples of processes which may cause charged particles to slow down without large angle scattering are elastic and inelastic scattering, ionization, Bremsstrahlung, and Čerenkov radiation.

In general, field geometric quantities: C, half bounce path length, I, and Φ remain adiabatically invariant with slowing down, but M, J, and K do not.

Acknowledgments

I am grateful to Dong Lin for the idea that C is invariant with slowing down and for comments on the criticism of the referee of the version for J.G.R. I thank all who read a preprint and made suggestions, particularly Tom Chang, Pradip Bakshi, and the above-mentioned referee. To the latter, among other things, is owed consideration of adiabatic invariants other than C and the invariance with slowing down of pure field geometric quantities.

References

1. Lin, D. L. (1980) Seminar on beam-plasma interactions in space, Air Force Geophysics Laboratory, 18 March 1980.
2. Lin, D. L. (1980) Adiabatic invariance in the presence of a slowing-down force, 26 March 1980, unpublished. Lin informed me that he first carried out this derivation 25 Oct. 1979. It will be included in a paper on a Monte Carlo code.
3. Dubs, C.W. (1980) The first adiabatic invariant for a particle slowing down, 26 March 1980, unpublished.
4. Alfvén, H., and Fälthammar, C.G. (1963) Cosmical Electrodynamics, Oxford University Press, London, pp. 29, 30.
5. Dubs, C.W. (1980) Adiabatic invariants for a charged particle slowing down, submitted to Journal of Geophysical Research.
6. Goldstein, H. (1950) Classical Mechanics, Addison-Wesley Press, Inc. p. 22.

Printed by
United States Air Force
Hanscom AFB, Mass. 01731

ATE

LMED

-8